Exercises for Stochastic Processes

Tutorial exercises:

Let B be a standard Brownian motion.

T1. Find stopping times σ and τ with $\mathbb{E}[\sigma] < \infty$, $\sigma \le \tau$ almost surely and

$$\mathbb{E}[B_\tau^2] < \mathbb{E}[B_\sigma^2].$$

T2. Let $f : \mathbb{R} \to \mathbb{R}$ be bounded and twice continuously differentiable with bounded first derivative and suppose that for all $t > 0$ and all $x \in \mathbb{R}$ we have $\mathbb{E}^x|f(B_t)| < \infty$ and $\mathbb{E}^x[\int_0^t |f''(B_s)|\mathrm{d}s] < \infty$. Show that the process defined by

$$X_t := f(B_t) - \frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s$$

is a martingale.

(Hint: The normal density $p_t(x,y) = \frac{1}{\sqrt{2\pi t}} \exp\big(-(x-y)^2/2t\big)$ satisfies the differential equation $\frac{\partial}{\partial t} p_t = \frac{1}{2}\frac{\partial^2}{\partial y^2} p_t$.)

T3. Show that, for any continuous time Markov chain with starting point $x \in S$, the time of the first jump has an exponential distribution (possibly with parameter 0 or ∞).

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Homework exercises:

Let B be a standard Brownian motion.

H1. Let μ be a probability distribution with mass on only three values $-a < 0 < b < c$ and mean zero, consider

$$\tau_s := \min\big\{ \inf\{t \geq 0 \mid B_t = -a\}, \inf\{t \geq s \mid B_t = b\}, \inf\{t \geq 0 \mid B_t = c\}\big\}.$$

Show that the distribution of B_{τ_s} varies continuously from the one on $\{-a, b\}$ with mean zero to the one on $\{-a, c\}$ if s is varied from 0 to ∞ and conclude that, for some $s \geq 0$, B_{τ_s} has distribution μ.

H2. Let $(X_t)_{0 \leq t \leq 1}$ be a Brownian bridge:

$$X_t := B_t - tB_1.$$

(a) Show that for all $x \geq 0$:

$$\lim_{\varepsilon \to 0} \mathbb{P}\left(\sup_{t \in [0,1]} B_t > x \,\middle|\, |B_1| \leq \varepsilon \right) = \mathbb{P}\left(\sup_{t \in [0,1]} X_t > x \right).$$

(Hint: Recall H3(b) of sheet 3.)

(b) Show that for all $x \geq 0$:

$$\mathbb{P}\left(\sup_{t \in [0,1]} X_t > x \right) = \exp\big(-2x^2 \big).$$

H3. Let $(X_i)_{i \in \mathbb{N}_0}$ be i.i.d. with mean 0 and variance 1, and let $S_k = \sum_{i=0}^k X_i$ for $k \in \mathbb{N}$. Show that

$$\lim_{n \to \infty} \mathbb{P}\left(\frac{1}{n} \max\{k \leq n \mid S_k S_{k+1} \leq 0\} \leq t \right) = \frac{2}{\pi} \arcsin \sqrt{t}$$

for $0 \leq t \leq 1$.
(Hint: Recall that the same arcsine distribution solved problem T3(b) on sheet 4.)

Deadline: Monday, 25.11.19